*Note: Jacob Warner and I did read over each other’s proofs and therefore worked together on a portion of this homework.*

**1. Write the best proof that you can for the proposition for Bad Proof #2 (see Supplement 1: Bad Proofs).**

Let be non-empty, bounded-above subsets of real numbers. Based on the definition of the union of two sets , or {. Additionally, we know that a set is bounded above if such that We need to prove that the union of two sets bounded-above are still bounded-above. By the definition of what it means for a set to be bounded above, we need to prove the existence of an element in such that the element is greater than or equal to any other element in which is defined as the union of and . Since A itself is bounded above such that . Similarly, since itself is also bounded above. Then, if , we know is bigger than or equal to any element in A or B. To prove that this is true, let us consider all cases.

Case I: Suppose that and assume that was not greater than any element in . We would then have . If , then , and we would reach the inequality , which is a contradiction. cannot simultaneously be less than or equal to and greater than . If , then , and we would reach the inequality , which is a contradiction. cannot less than or equal to and strictly greater than when

Case II: Suppose that and assume that was not greater than any element in . We would then have . If , then , and we would reach the inequality , which is a contradiction. cannot less than or equal to and strictly greater than when . If , then , and we would reach the inequality , which is a contradiction. cannot simultaneously be less than or equal to and greater than .

We reach a contradiction when trying to show that there exists an element or such that, if , Therefore, Equivalently, the union of and is bounded above by

**2. Find examples of , where , such that dividing into gives the same quotient and remainder as dividing and d into c, but where**

* , quotient = 5 & remainder = 0.
* , quotient = 3 & remainder = 0.
* , quotient = 3 & remainder = 3.

**3. Let and suppose that dividing intoresults in the quotient with remainder .**

1. *Conjecture:* Since division is based on repeated subtraction at its root, dividing into can be replicated through repeated subtraction in “Proof by Procedure”, which will also yield the solution in the form of a unique quotient and remainder. We can let and while also rearranging to solve for and get . Since we don’t know the unique value of that satisfies the equation, we can replace it with an integer and let it vary. So,

When we reach an such that we will know that we found the unique and smallest value of that satisfies the equation. Additionally, the number of subtractions we do, represented in the value of integer , will reflect the unique quotient value,

Example 1:

15

9

3

We see that satisfies the restriction on such that Therefore, we have found our unique

Example 2:

We see that satisfies the restriction on such that Therefore, we have found our unique

Example 3:

We see that satisfies the restriction on such that Therefore, we have found our unique

1. Let , and let . Recall that the Difference Algorithm says that there exist unique integers such that for which follows from If we then consider , then we want to show how to find the such that . In the case of we can let and let to fit the Division Algorithm exactly. We can rearrange the Division Algorithm so that repeated subtraction will achieve the same purpose as division to get Then, the set

will have a smallest element per the well-ordering principle of the integers. We stated earlier that and , so we can substitute those into our set builder notation above to get

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The smallest element in the set will represent the value of , and the number of elements in the set will represent the value of This set now includes a way to find unique integers using integers with the division algorithm.

**4. Prove that if are sets, then**

Let , where is the universal set. For any element , we say x is in the union of two sets , or {. . Further, for any element , we say x is in the intersection of two sets or To prove set equivalency of two sets we must show that and A. Note that we will refer to as LHS (Left Hand Side) and ( as RHS (Right Hand Side).

First, let us consider the set . We do not care about any element because both sides’ truth values are contingent upon because of the intersections, which say that must be in two sets simultaneously to be in the intersection of those two sets. Thus, if cannot be in the intersection involving any set and , so both the LHS and RHS evaluate to false.

Next, let us consider the sets and . Based on our earlier assumptions with we will assume that from now on. The truth value of the both LHS and RHS is contingent upon or because of the union. Thus, the only time when both LHS and RHS evaluate to false is when and simultaneously.

Finally, we will look at the truth values of the LHS and RHS and their equivalency. We already noted that if , both LHS and RHS are simultaneously false. Each is contingent upon because of the intersections involving . Next, both LHS and RHS are simultaneously true in the following three scenarios:

and

and ,

3. and .

Both LHS and RHS involve the union of and , which means that an element also be in either or for the truth value to hold. The only time that both LHS and RHS are simultaneously false is when and We have shown that LHS and RHS are either simultaneously true or simultaneously false, which is another way of saying and , proving that

**5. Use the properties of Z that we discussed in class to prove that if , then .**

Suppose be chosen arbitrarily. We will reference the proof from class: . We want to show that . Note that whenever we say reference a property of the integers, we will simply say “property x” instead of stating “property x of the integers.” To begin our proof, we start out with . From the transitive property, we can multiply by an integer on both sides of the equation, which yields Per property 11, Per property 8, . Per property 10, Per property 3, Per property 5, additive inverses exist in the integers. In this case, must be the additive inverse of because the two added together equal In other words, we see that for the addition of the two terms to equal zero, they must be equal in magnitude but opposite in sign. If were to be set equal to , it’s sign must be switched. It therefore holds that .

**6. Let and be integers and assume that . By the division algorithm, there are unique integers and such that and .**

**Prove that divides if and only if .**

Let where for any there exist unique such that . To show divides if and only if means that we must show divides and divides We must also consider the definition of divisibility of two numbers . We say divides if for some .

First, let us assume that divides and try to deduce that Following our assumption that divides , let us assume that From the definition of division, we know that for some since divides Thus, per the Division Algorithm, we have

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We can then rearrange the equation to solve for . Additionally, we stated above that , so we can substitute that in accordingly to get

.

We can factor out a , and from the restrictions on , we have

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We can then take away throughout the inequality, and we get

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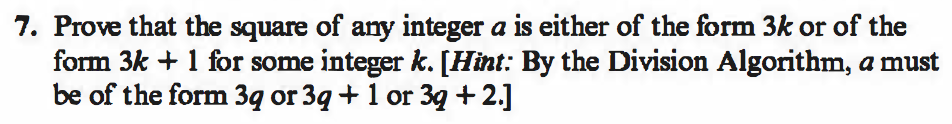
Subtraction isn’t closed under addition, but we can use the fact that and that the integers are not dense to derive that must be equal to . Remember, we initially substituted into the restriction, which was equivalent to . Thus, So, by property 11 of the integers. Thus, we see that if divides , then

Second, let us assume that and try to deduce that divides . If we have

By property 4 of , the previous equation results in the following equation:

,

which fits the definition of divides . We already noted that , so it is a whole number. Thus, if , then divides . This wraps up our proof because we showed both divides and divides . Thus, we have shown that divides .



The general form of the Division Algorithm states that such that . So, this problem is a special case of the Division Algorithm such that Thus, there must exist unique integers such that . We therefore have the 3 separate cases: or . So, must be in the form of , and for some To prove that the square of is either in the form of or for some integer we will look at each case separately.

: Suppose that a= Then, we have the following:

,

Case : Suppose that a= Then, we have the following:

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Case : Suppose that a= Then, we have the following:

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We see that the square of for some results in the form of either or for some integer .

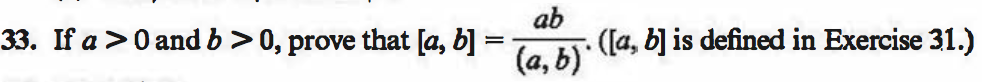
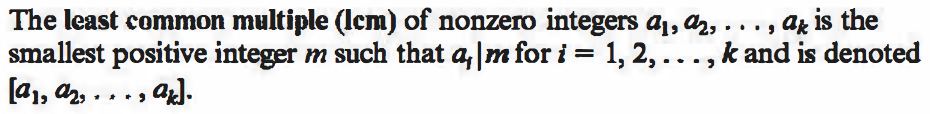
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This proposition is essentially saying that 1, the GCD of (), is a linear combination of and . Theorem 1.2 from the book says we can verify this by showing the existence of integers such that . Note that the integers may not be unique. We can satisfy the proof by finding unique values of that satisfy the linear combination. In this case, and satisfy the equation:

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Thus, for any .

*Challenge Problem*



Let denote the , and let denote the . We want to show that for and We can rewrite the proposition equation, per the definition of divisibility, which says that for some , such that By multiplying by we can get Similarly, to proving set equivalence, if and only if

and

First, note that of all common multiples of two integers , the least common multiple is the smallest of all common multiples. Thus, for any common multiple , there exists a common multiple such that or . We will refer to this inequality as .

Next, note that of all common denominator of two integers , the greatest common denominator is the greatest of all common denominators. Thus, for any common denominator , there exists a common denominator such that or . We will refer to this inequality as

By combining and we get , or by rearranging algebraically, we get Similarly, we have .